# On the Asymptotic Expansion of the Solutions of the Separated Nonlinear Schrödinger Equation

A. A. Kapaev,

*St Petersburg Department of Steklov Mathematical Institute,*
*Fontanka 27, St Petersburg 191011, Russia,*

V. E. Korepin,

*C.N.Yang Institute for Theoretical Physics,*
*State University of New York at Stony Brook,*
*Stony Brook, NY 11794-3840, USA*

**Abstract**

Nonlinear Schrödinger equation with the Schwarzian initial data is important in nonlinear optics, Bose condensation and in the theory of strongly correlated electrons. The asymptotic solutions in the region $x/t = \mathcal{O}(1)$, $t \to \infty$, can be represented as a double series in $t^{-1}$ and $\ln t$. Our current purpose is the description of the asymptotics of the coefficients of the series.

## 1 Introduction

A coupled nonlinear dispersive partial differential equation in $(1+1)$ dimension for the functions $g_+$ and $g_-$,

$$
-i\partial_t g_+ = \frac{1}{2}\partial_x^2 g_+ + 4g_+^2 g_-,
$$
$$
i\partial_t g_- = \frac{1}{2}\partial_x^2 g_- + 4g_-^2 g_+, \tag{1}
$$

called the separated Nonlinear Schrödinger equation (sNLS), contains the conventional NLS equation in both the focusing and defocusing forms as $g_+ = \bar{g}_-$ or $g_+ = -\bar{g}_-$, respectively. For certain physical applications, e.g. in nonlinear optics, Bose condensation, theory of strongly correlated electrons, see [1] – [9], the detailed information on the long time asymptotics of solutions with initial conditions rapidly decaying as $x \to \pm\infty$ is quite useful for qualitative explanation of the experimental phenomena.

Our interest to the long time asymptotics for the sNLS equation is inspired by its application to the Hubbard model for one-dimensional gas of strongly correlated electrons. The model explains a remarkable effect of charge and spin separation, discovered experimentally by C. Kim, Z.-X.M. Shen, N. Motoyama, H. Eisaki, S. Ushida, T. Tohyama and S. Maekawa [19]. Theoretical justification

1

of the charge and spin separation include the study of temperature dependent correlation functions in the Hubbard model. In the papers [1]–[3], it was proven that time and temperature dependent correlations in Hubbard model can be described by the sNLS equation (1).

For the systems completely integrable in the sense of the Lax representation [10, 11], the necessary asymptotic information can be extracted from the Riemann-Hilbert problem analysis [12]. Often, the fact of integrability implies the existence of a long time expansion of the generic solution in a formal series, the successive terms of which satisfy some recurrence relation, and the leading order coefficients can be expressed in terms of the spectral data for the associated linear system. For equation (1), the Lax pair was discovered in [13], while the formulation of the Riemann-Hilbert problem can be found in [8]. As $t \to \infty$ for $x/t$ bounded, system (1) admits the formal solution given by

$$g_+ = e^{i\frac{x^2}{2t} - (\frac{1}{2} + i\nu) \ln 4t} \Big( u_0 + \sum_{n=1}^{\infty} \sum_{k=0}^{2n} \frac{(\ln 4t)^k}{t^n} u_{nk} \Big),$$

$$g_- = e^{-i\frac{x^2}{2t} - (\frac{1}{2} - i\nu) \ln 4t} \Big( v_0 + \sum_{n=1}^{\infty} \sum_{k=0}^{2n} \frac{(\ln 4t)^k}{t^n} v_{nk} \Big), \qquad (2)$$

where the quantities $\nu$, $u_0$, $v_0$, $u_{nk}$ and $v_{nk}$ are some functions of $\lambda_0 = -x/2t$.

For the NLS equation ($g_+ = \pm \bar{g}_-$), the asymptotic expansion was suggested by M. Ablowitz and H. Segur [6]. For the defocusing NLS ($g_+ = -\bar{g}_-$), the existence of the asymptotic series (2) is proven by P. Deift and X. Zhou [9] using the Riemann-Hilbert problem analysis, and there is no principal obstacle to extend their approach for the case of the separated NLS equation. Thus we refer to (2) as the Ablowitz-Segur-Deift-Zhou expansion. Expressions for the leading coefficients for the asymptotic expansion of the conventional NLS equation in terms of the spectral data were found by S. Manakov, V. Zakharov, H. Segur and M. Ablowitz, see [14]–[16]. The general sNLS case was studied by A. Its, A. Izergin, V. Korepin and G. Varzugin [17], who have expressed the leading order coefficients $u_0$, $v_0$ and $\nu = -u_0 v_0$ in (2) in terms of the spectral data.

The generic solution of the focusing NLS equation contains solitons and radiation. The interaction of the single soliton with the radiation was described by Segur [18]. It can be shown that, for the generic Schwarzian initial data and generic bounded ratio $x/t$, $|c - \frac{x}{2t}| < M$, $c = const$, where $M$ is small enough, the expansion (2) for sNLS equation (1) represents the contribution of the continuous spectrum when possible solitons go away from the moving frame. Thus the soliton contribution in the considered asymptotic area is exponentially small for large $t$ and can be neglected in compare with contributions of the power-log terms.

Below, we pay our special attention to the coefficients $u_{n,2n}$, $v_{n,2n}$, which determine the leading behavior of $g_+$ and $g_-$ as $t \to \infty$ in the order $t^{-n}$. For

2

these coefficients as well as for $u_{n,2n-1}$, $v_{n,2n-1}$, we find simple *exact* formulae

$$u_{n,2n} = u_0 \frac{i^n (\nu')^{2n}}{8^n n!}, \quad v_{n,2n} = v_0 \frac{(-i)^n (\nu')^{2n}}{8^n n!}, \tag{3}$$

and (20) below. We describe coefficients at other powers of $\ln t$ using the generating functions which can be reduced to a system of polynomials satisfying the recursion relations, see (24), (23). As a by-product, we modify the Ablowitz-Segur-Deift-Zhou expansion (2),

$$g_+ = \exp\left[i\frac{x^2}{2t} - (\frac{1}{2} + i\nu)\ln 4t + i\frac{(\nu')^2 \ln^2 4t}{8t}\right] \sum_{n=0}^{\infty} \sum_{k=0}^{2n-[\frac{n+1}{2}]} \frac{(\ln 4t)^k}{t^n} \tilde{u}_{n,k},$$

$$g_- = \exp\left[-i\frac{x^2}{2t} - (\frac{1}{2} - i\nu)\ln 4t - i\frac{(\nu')^2 \ln^2 4t}{8t}\right] \sum_{n=0}^{\infty} \sum_{k=0}^{2n-[\frac{n+1}{2}]} \frac{(\ln 4t)^k}{t^n} \tilde{v}_{n,k}, \tag{4}$$

## 2   Recurrence relations and generating functions

Substituting (2) into (1), and equating coefficients of $t^{-1}$, we find

$$\nu = -u_0 v_0. \tag{5}$$

In the order $t^{-n}$, $n \geq 2$, equating coefficients of $\ln^j 4t$, $0 \leq j \leq 2n$, we obtain the recursion

$$-i(j+1)u_{n,j+1} + inu_{n,j} = \nu u_{n,j} - \frac{i\nu''}{8} u_{n-1,j-1} - \frac{(\nu')^2}{8} u_{n-1,j-2} -$$

$$-\frac{i\nu'}{4} u'_{n-1,j-1} + \frac{1}{8} u''_{n-1,j} + \sum_{\substack{l,k,m=0 \\ l+k+m=n}}^{n} \sum_{\substack{\alpha=0,\ldots,2l \\ \beta=0,\ldots,2k \\ \gamma=0,\ldots,2m \\ \alpha+\beta+\gamma=j}} u_{l,\alpha} u_{k,\beta} v_{m,\gamma}, \tag{6}$$

$$i(j+1)v_{n,j+1} - inv_{n,j} = \nu v_{n,j} + \frac{i\nu''}{8} v_{n-1,j-1} - \frac{(\nu')^2}{8} v_{n-1,j-2} +$$

$$+\frac{i\nu'}{4} v'_{n-1,j-1} + \frac{1}{8} v''_{n-1,j} + \sum_{\substack{l,k,m=0 \\ l+k+m=n}}^{n} \sum_{\substack{\alpha=0,\ldots,2l \\ \beta=0,\ldots,2k \\ \gamma=0,\ldots,2m \\ \alpha+\beta+\gamma=j}} u_{l,\alpha} v_{k,\beta} v_{m,\gamma}, \tag{7}$$

where the prime means differentiation with respect to $\lambda_0 = -x/(2t)$.

*Master generating functions* $F(z,\zeta)$, $G(z,\zeta)$ for the coefficients $u_{n,k}$, $v_{n,k}$ are defined by the formal series

$$F(z,\zeta) = \sum_{n,k} u_{n,k} z^n \zeta^k, \quad G(z,\zeta) = \sum_{n,k} v_{n,k} z^n \zeta^k, \tag{8}$$

3

where the coefficients $u_{n,k}$, $v_{n,k}$ vanish for $n < 0$, $k < 0$ and $k > 2n$. It is straightforward to check that the master generating functions satisfy the nonstationary separated Nonlinear Schrödinger equation in $(1+2)$ dimensions,

$$-iF_\zeta + izF_z = \left(\nu - \frac{i\nu''}{8}z\zeta - \frac{(\nu')^2}{8}z\zeta^2\right)F - \frac{i\nu'}{4}z\zeta F' + \frac{1}{8}zF'' + F^2G,$$

$$iG_\zeta - izG_z = \left(\nu + \frac{i\nu''}{8}z\zeta - \frac{(\nu')^2}{8}z\zeta^2\right)G + \frac{i\nu'}{4}z\zeta G' + \frac{1}{8}zG'' + FG^2. \quad (9)$$

We also consider the sectional generating functions $f_j(z)$, $g_j(z)$, $j \geq 0$,

$$f_j(z) = \sum_{n=0}^\infty u_{n,2n-j}z^n, \quad g_j(z) = \sum_{n=0}^\infty v_{n,2n-j}z^n. \quad (10)$$

Note, $f_j(z) \equiv g_j(z) \equiv 0$ for $j < 0$ because $u_{n,k} = v_{n,k} = 0$ for $k > 2n$. The master generating functions $F$, $G$ and the sectional generating functions $f_j$, $g_j$ are related by the equations

$$F(z\zeta^{-2}, \zeta) = \sum_{j=0}^\infty \zeta^{-j}f_j(z), \quad G(z\zeta^{-2}, \zeta) = \sum_{j=0}^\infty \zeta^{-j}g_j(z). \quad (11)$$

Using (11) in (9) and equating coefficients of $\zeta^{-j}$, we obtain the differential system for the sectional generating functions $f_j(z)$, $g_j(z)$,

$$-2iz\partial_z f_{j-1} + i(j-1)f_{j-1} + iz\partial_z f_j =$$

$$= \nu f_j - z\frac{i\nu''}{8}f_{j-1} - z\frac{(\nu')^2}{8}f_j - z\frac{i\nu'}{4}f'_{j-1} + z\frac{1}{8}f''_{j-2} + \sum_{\substack{k,l,m=0 \\ k+l+m=j}}^j f_k f_l g_m,$$

$$2iz\partial_z g_{j-1} - i(j-1)g_{j-1} - iz\partial_z g_j = \qquad (12)$$

$$= \nu g_j + z\frac{i\nu''}{8}g_{j-1} - z\frac{(\nu')^2}{8}g_j + z\frac{i\nu'}{4}g'_{j-1} + z\frac{1}{8}g''_{j-2} + \sum_{\substack{k,l,m=0 \\ k+l+m=j}}^j f_k g_l g_m.$$

Thus, the generating functions $f_0(z)$, $g_0(z)$ for $u_{n,2n}$, $v_{n,2n}$ solve the system

$$iz\partial_z f_0 = \nu f_0 - z\frac{(\nu')^2}{8}f_0 + f_0^2 g_0, \quad -iz\partial_z g_0 = \nu g_0 - z\frac{(\nu')^2}{8}g_0 + f_0 g_0^2. \quad (13)$$

The system implies that the product $f_0(z)g_0(z) \equiv const$. Since $f_0(0) = u_0$ and $g_0(0) = v_0$, we obtain the identity

$$f_0 g_0(z) = -\nu. \quad (14)$$

Using (14) in (13), we easily find

$$f_0(z) = u_0 e^{i\frac{(\nu')^2}{8}z} = u_0 \sum_{n=0}^\infty \frac{i^n(\nu')^{2n}}{8^n n!}z^n,$$

4

$$g_0(z) = v_0 e^{-i\frac{(\nu')^2}{8}z} = v_0 \sum_{n=0}^{\infty} \frac{(-i)^n (\nu')^{2n}}{8^n n!} z^n, \qquad (15)$$

which yield the explicit expressions (3) for the coefficients $u_{n,2n}$, $v_{n,2n}$.

Generating functions $f_1(z)$, $g_1(z)$ for $u_{n,2n-1}$, $v_{n,2n-1}$, satisfy the differential system

$$-2iz\partial_z f_0 + iz\partial_z f_1 = \nu f_1 - z\frac{i\nu''}{8}f_0 - z\frac{(\nu')^2}{8}f_1 - z\frac{i\nu'}{4}f_0' + 2f_1 f_0 g_0 + f_0^2 g_1,$$

$$2iz\partial_z g_0 - iz\partial_z g_1 = \nu g_1 + z\frac{i\nu''}{8}g_0 - z\frac{(\nu')^2}{8}g_1 + z\frac{i\nu'}{4}g_0' + f_1 g_0^2 + 2f_0 g_0 g_1. \ (16)$$

We will show that the differential system (16) for $f_1(z)$ and $g_1(z)$ is solvable in terms of elementary functions. First, let us introduce the auxiliary functions

$$p_1(z) = \frac{f_1(z)}{f_0(z)}, \quad q_1(z) = \frac{g_1(z)}{g_0(z)}.$$

These functions satisfy the non-homogeneous system of linear ODEs

$$\partial_z p_1 = \frac{i\nu}{z}(p_1 + q_1) + \frac{i(\nu')^2}{4} - \frac{\nu''}{8} - \frac{\nu'}{4}\frac{f_0'}{f_0},$$

$$\partial_z q_1 = -\frac{i\nu}{z}(p_1 + q_1) - \frac{i(\nu')^2}{4} - \frac{\nu''}{8} - \frac{\nu'}{4}\frac{g_0'}{g_0}, \qquad (17)$$

so that

$$\partial_z(q_1 + p_1) = -\frac{(\nu^2)''}{8\nu}, \qquad (18)$$

where we have used the relations (14) and (15). Taking into account the initial condition $p_1(0) = q_1(0) = 0$, the system (17) is easily integrated and we arrive at the following expressions for the generating functions $f_1$, $g_1$,

$$f_1(z) = p_1(z)f_0(z), \quad f_0(z) = u_0 e^{i\frac{(\nu')^2}{8}z},$$

$$p_1(z) = \left(-\frac{i\nu\nu''}{4} - \frac{\nu''}{8} - \frac{\nu'u_0'}{4u_0}\right)z - i\frac{(\nu')^2\nu''}{32}z^2,$$

$$g_1(z) = q_1(z)g_0(z), \quad g_0(z) = v_0 e^{-i\frac{(\nu')^2}{8}z},$$

$$q_1(z) = \left(\frac{i\nu\nu''}{4} - \frac{\nu''}{8} - \frac{\nu'v_0'}{4v_0}\right)z + i\frac{(\nu')^2\nu''}{32}z^2. \qquad (19)$$

Finally, expanding the generating functions in series of $z$, we find the coefficients $u_{n,2n-1}$ and $v_{n,2n-1}$,

$$u_{1,1} = u_0\left(-\frac{i\nu\nu''}{4} - \frac{\nu''}{8} - \frac{\nu'u_0'}{4u_0}\right), \quad v_{1,1} = v_0\left(\frac{i\nu\nu''}{4} - \frac{\nu''}{8} - \frac{\nu'v_0'}{4v_0}\right), \qquad (20)$$

$$u_{n,2n-1} = -2u_0 \frac{i^{n-1}(\nu')^{2(n-1)}}{8^n(n-2)!}\left(\nu'' + \frac{1}{n-1}\left(\frac{\nu''}{2} + i\nu\nu'' + \frac{\nu'u_0'}{u_0}\right)\right), \quad n \ge 2,$$

$$v_{n,2n-1} = -2v_0 \frac{(-i)^{n-1}(\nu')^{2(n-1)}}{8^n(n-2)!}\left(\nu'' + \frac{1}{n-1}\left(\frac{\nu''}{2} - i\nu\nu'' + \frac{\nu'v_0'}{v_0}\right)\right), \quad n \ge 2.$$

Generating functions $f_j(z)$, $g_j(z)$ for $u_{n,2n-j}$, $v_{n,2n-j}$, $j \ge 2$, satisfy the differential system (12). Similarly to the case $j = 1$ above, let us introduce the auxiliary functions $p_j$ and $q_j$,

$$p_j = \frac{f_j}{f_0}, \quad q_j = \frac{g_j}{g_0}. \tag{21}$$

In the terms of these functions, the system (12) reads,

$$\partial_z p_j = \frac{i\nu}{z}(p_j + q_j) + a_j, \quad \partial_z q_j = -\frac{i\nu}{z}(p_j + q_j) + b_j, \tag{22}$$

where

$$a_j = 2\partial_z p_{j-1} + \left(\frac{i(\nu')^2}{4} - \frac{\nu''}{8} - \frac{j-1}{z}\right)p_{j-1} -$$

$$-\frac{\nu'}{4}\frac{(p_{j-1}f_0)'}{f_0} - i\frac{(p_{j-2}f_0)''}{8f_0} + \frac{i\nu}{z}\sum_{\substack{k,l,m=0 \\ k+l+m=j}}^{j-1} p_k p_l q_m,$$

$$b_j = 2\partial_z q_{j-1} + \left(-\frac{i(\nu')^2}{4} - \frac{\nu''}{8} - \frac{j-1}{z}\right)q_{j-1} -$$

$$-\frac{\nu'}{4}\frac{(q_{j-1}g_0)'}{g_0} + i\frac{(q_{j-2}g_0)''}{8g_0} - \frac{i\nu}{z}\sum_{\substack{k,l,m=0 \\ k+l+m=j}}^{j-1} p_k q_l q_m. \tag{23}$$

With the initial condition $p_j(0) = q_j(0) = 0$, the system is easily integrated and uniquely determines the functions $p_j(z)$, $q_j(z)$,

$$p_j(z) = \int_0^z a_j(\zeta)\,d\zeta + i\nu \int_0^z \frac{d\zeta}{\zeta}\int_0^\zeta d\xi(a_j(\xi) + b_j(\xi)),$$

$$q_j(z) = \int_0^z b_j(\zeta)\,d\zeta - i\nu \int_0^z \frac{d\zeta}{\zeta}\int_0^\zeta d\xi(a_j(\xi) + b_j(\xi)). \tag{24}$$

These equations with expressions (23) together establish the recursion relation for the functions $p_j(z)$, $q_j(z)$. In terms of $p_j(z)$ and $q_j(z)$, expansion (2) reads

$$g_+ = e^{i\frac{x^2}{2t} - (\frac{1}{2}+i\nu)\ln 4t + i\frac{(\nu')^2\ln^2 4t}{8t}} u_0 \sum_{j=0}^\infty \frac{p_j\left(\frac{\ln^2 4t}{t}\right)}{\ln^j 4t},$$

$$g_- = e^{-i\frac{x^2}{2t} - (\frac{1}{2}-i\nu)\ln 4t - i\frac{(\nu')^2\ln^2 4t}{8t}} v_0 \sum_{j=0}^\infty \frac{q_j\left(\frac{\ln^2 4t}{t}\right)}{\ln^j 4t}. \tag{25}$$

Let $a_j(z)$ and $b_j(z)$ be polynomials of degree $M$ with the zero $z = 0$ of multiplicity $m$,

$$a_j(z) = \sum_{k=m}^{M} a_{jk} z^k, \quad b_j(z) = \sum_{k=m}^{M} b_{jk} z^k.$$

Then the functions $p_j(z)$ and $q_j(z)$ (24) are polynomials of degree $M + 1$ with a zero at $z = 0$ of multiplicity $m + 1$,

$$p_j(z) = \sum_{k=m+1}^{M+1} \frac{1}{k}\big(a_{j,k-1} + \frac{i\nu}{k}(a_{j,k-1} + b_{j,k-1})\big) z^k,$$

$$q_j(z) = \sum_{k=m+1}^{M+1} \frac{1}{k}\big(b_{j,k-1} - \frac{i\nu}{k}(a_{j,k-1} + b_{j,k-1})\big) z^k. \tag{26}$$

On the other hand, $a_j(z)$ and $b_j(z)$ are described in (23) as the actions of the differential operators applied to the functions $p_{j'}$, $q_{j'}$ with $j' < j$. Because $p_0(z) = q_0(z) \equiv 1$ and $p_1(z)$, $q_1(z)$ are polynomials of the second degree and a single zero at $z = 0$, cf. (19), it easy to check that $a_2(z)$ and $b_2(z)$ are non-homogeneous polynomials of the third degree such that

$$a_{2,3} = -\frac{(\nu')^4 (\nu'')^2}{2^{10}}(2 - i\nu), \quad b_{2,3} = -\frac{(\nu')^4 (\nu'')^2}{2^{10}}(2 + i\nu), \tag{27}$$

$$a_{2,0} = -\frac{i\nu\nu''}{4} - \frac{\nu''}{8} - \frac{\nu' u_0'}{4u_0} - \frac{i u_0''}{8u_0}, \quad b_{2,0} = \frac{i\nu\nu''}{4} - \frac{\nu''}{8} - \frac{\nu' v_0'}{4v_0} + \frac{i v_0''}{8v_0}.$$

Thus $p_2(z)$ and $q_2(z)$ are polynomials of the fourth degree with a single zero at $z = 0$. Some of their coefficients are

$$p_{2,4} = q_{2,4} = -\frac{(\nu')^4 (\nu'')^2}{2^{11}}, \tag{28}$$

$$p_{2,1} = -\frac{i(\nu')^2}{4} - (1 + 2i\nu)\big(\frac{\nu''}{4} + \frac{i u_0''}{8u_0}\big) - \frac{\nu(u_0')^2}{4u_0^2},$$

$$q_{2,1} = \frac{i(\nu')^2}{4} - (1 - 2i\nu)\big(\frac{\nu''}{4} - \frac{i v_0''}{8v_0}\big) - \frac{\nu(v_0')^2}{4v_0^2}.$$

**Lemma 2.1** *Functions $p_j(z)$, $q_j(z)$, $j \in \mathbb{Z}_+$, defined in (21) are polynomials of degree not exceeding $2j$. The zero of $p_j(z)$ and $q_j(z)$ at $z = 0$ has multiplicity no less that $m_j = \left[\frac{j+1}{2}\right]$.*

*Proof.* The assertion holds true for $j = 0, 1, 2$. Let it be correct for $\forall j < j'$. Then $a_{j'}(z)$ and $b_{j'}(z)$ are defined as the sum of polynomials. The maximal degrees of such polynomials are $\deg\big((p_{j'-1}f_0)'/f_0\big) = 2j' - 1$, $\deg\big((q_{j'-1}g_0)'/g_0\big) =$

$2j' - 1$, and

$$\deg\Big(\frac{1}{z} \sum_{\substack{\alpha,\beta,\gamma=0 \\ \alpha+\beta+\gamma=j'}}^{j'-1} p_\alpha p_\beta q_\gamma\Big) = 2j' - 1, \quad \deg\Big(\frac{1}{z} \sum_{\substack{\alpha,\beta,\gamma=0 \\ \alpha+\beta+\gamma=j'}}^{j'-1} p_\alpha q_\beta q_\gamma\Big) = 2j' - 1.$$

Thus $\deg a_{j'}(z) = \deg b_{j'}(z) \le 2j' - 1$, and $\deg p_{j'}(z) = \deg q_{j'}(z) \le 2j'$.

Multiplicity of the zero at $z = 0$ of $a_{j'}(z)$ and $b_{j'}(z)$ is no less than the minimal multiplicity of the summed polynomials in (23), but the minor coefficients of the polynomials $2\partial_z p_{j'-1}$ and $-(j-1)p_{j'-1}/z$, as well as of $2\partial_z q_{j'-1}$ and $-(j-1)q_{j'-1}/z$ may cancel each other. Let $j' = 2k$ be even. Then

$$m_{j'} = \min\big\{m_{j'-1}; \ m_{j'-2} + 1; \min_{\substack{\alpha,\beta,\gamma=0,\ldots,j'-1 \\ \alpha+\beta+\gamma=j'}} \big(m_\alpha + m_\beta + m_\gamma\big)\big\} = \frac{j'}{2} = [\frac{j'+1}{2}].$$

Let $j' = 2k - 1$ be odd. Then $2m_{j'-1} - (j'-1) = 0$, and

$$m_{j'} = \min\big\{m_{j'-1} + 1; \ m_{j'-2} + 1; \min_{\substack{\alpha,\beta,\gamma=0,\ldots,j'-1 \\ \alpha+\beta+\gamma=j'}} \big(m_\alpha + m_\beta + m_\gamma\big)\big\} = \frac{j'+1}{2}.$$

$\square$

We arrive at the following expansions of the polynomials $p_j(z)$, $q_j(z)$,

$$p_j(z) = \sum_{k=[\frac{j+1}{2}]}^{2j} p_{j,k} z^k, \quad q_j(z) = \sum_{k=[\frac{j+1}{2}]}^{2j} q_{j,k} z^k. \tag{29}$$

The use of (29) in (25) yields (4). On the other hand, using (29) in (21), we obtain the expressions for the generating functions $f_j(z)$, $g_j(z)$,

$$f_j(z) = u_0 \sum_{n=[\frac{j+1}{2}]}^{\infty} z^n \sum_{k=\max\{0;n-2j\}}^{n-[\frac{j+1}{2}]} p_{j,n-k} \frac{i^k (\nu')^{2k}}{8^k k!},$$

$$g_j(z) = v_0 \sum_{n=[\frac{j+1}{2}]}^{\infty} z^n \sum_{k=\max\{0;n-2j\}}^{n-[\frac{j+1}{2}]} q_{j,n-k} \frac{(-i)^k (\nu')^{2k}}{8^k k!}, \tag{30}$$

and the coefficients of our interest,

$$u_{n,2n-j} = u_0 \sum_{k=\max\{0;n-2j\}}^{n-[\frac{j+1}{2}]} p_{j,n-k} \frac{i^k (\nu')^{2k}}{8^k k!},$$

$$v_{n,2n-j} = v_0 \sum_{k=\max\{0;n-2j\}}^{n-[\frac{j+1}{2}]} q_{j,n-k} \frac{(-i)^k (\nu')^{2k}}{8^k k!}. \tag{31}$$

In particular, the leading asymptotic term of these coefficients as $n \to \infty$ and $j$ fixed is given by

$$u_{n,2n-j} = u_0 p_{j,2j} \frac{i^{n-2j}(\nu')^{2(n-2j)}}{8^{n-2j}(n-2j)!}\left(1 + \mathcal{O}(\frac{1}{n})\right),$$

$$v_{n,2n-j} = v_0 q_{j,2j} \frac{(-i)^{n-2j}(\nu')^{2(n-2j)}}{8^{n-2j}(n-2j)!}\left(1 + \mathcal{O}(\frac{1}{n})\right). \tag{32}$$

Thus we have reduced the problem of the evaluation of the asymptotics of the coefficients $u_{n,2n-j}$ $v_{n,2n-j}$ for large $n$ to the computation of the leading coefficients of the polynomials $p_j(z)$, $q_j(z)$. In fact, using (24) or (26) and (23), it can be shown that the coefficients $p_{j,2j}$, $q_{j,2j}$ satisfy the recurrence relations

$$p_{j,2j} = -i\frac{(\nu')^2\nu''}{2^5 j}p_{j-1,2(j-1)} + \frac{i\nu}{2j}\sum_{\substack{k,l,m=0 \\ k+l+m=j}}^{j-1} p_{k,2k}p_{l,2l}q_{m,2m} +$$

$$+\frac{\nu(\nu')^2\nu''}{2^6 j^2}\left(p_{j-1,2(j-1)} - q_{j-1,2(j-1)}\right) - \frac{\nu^2}{4j^2}\sum_{\substack{k,l,m=0 \\ k+l+m=j}}^{j-1} p_{k,2k}\left(p_{l,2l} - q_{l,2l}\right)q_{m,2m},$$

$$q_{j,2j} = i\frac{(\nu')^2\nu''}{2^5 j}q_{j-1,2(j-1)} - \frac{i\nu}{2j}\sum_{\substack{k,l,m=0 \\ k+l+m=j}}^{j-1} p_{k,2k}q_{l,2l}q_{m,2m} - \tag{33}$$

$$-\frac{\nu(\nu')^2\nu''}{2^6 j^2}\left(p_{j-1,2(j-1)} - q_{j-1,2(j-1)}\right) + \frac{\nu^2}{4j^2}\sum_{\substack{k,l,m=0 \\ k+l+m=j}}^{j-1} p_{k,2k}\left(p_{l,2l} - q_{l,2l}\right)q_{m,2m}.$$

Similarly, the coefficients $u_{n,0}$, $v_{n,0}$ for the non-logarithmic terms appears from (31) for $j = 2n$, and are given simply by

$$u_{n,0} = u_0 p_{2n,n}, \quad v_{n,0} = v_0 q_{2n,n}. \tag{34}$$

Thus the problem of evaluation of the asymptotics of the coefficients $u_{n,0}$, $v_{n,0}$ for $n$ large is equivalent to computation of the asymptotics of the minor coefficients in the polynomials $p_j(z)$, $q_j(z)$. However, the last problem does not allow a straightforward solution because, according to (8), the sectional generating functions for the coefficients $u_{n,0}$, $v_{n,0}$ are given by

$$F(z,0) = \sum_{n=0}^{\infty} u_{n,0}z^n, \quad G(z,0) = \sum_{n=0}^{\infty} v_{n,0}z^n,$$

and solve the separated Nonlinear Schrödinger equation

$$-iF_\zeta + izF_z = \nu F + \frac{1}{8}zF'' + F^2G,$$

$$iG_\zeta - izG_z = \nu G + \frac{1}{8}zG'' + FG^2. \tag{35}$$

9

# 3  Discussion

Our consideration based on the use of generating functions of different types reveals the asymptotic behavior of the coefficients $u_{n,2n-j}$, $v_{n,2n-j}$ as $n \to \infty$ and $j$ fixed for the long time asymptotic expansion (2) of the generic solution of the sNLS equation (1). The leading order dependence of these coefficients on $n$ is described by the ratio $\frac{a^n}{(n-2j)!}$.

However, the behavior of the coefficients $u_{n,k}$, $v_{n,k}$ corresponding to other directions in $(n,k)$-plane can be dramatically different. For instance, the generating functions for $u_{n,0}$, $v_{n,0}$ solving equation (35) are related to the so-called isomonodromy solutions [20] of the system (1). These solutions are characterized by the formal asymptotic expansion (2) without log-terms. Actually, it takes place when parameter $\nu = -u_0 v_0$ in (2) is independent of $\lambda_0 = -x/2t$. Concerning the asymptotic behavior of the coefficients $u_{n,0}$, $v_{n,0}$ as $n \to \infty$, it is the plausible conjecture that $u_{n,0}, v_{n,0} \sim c^n \Gamma(\frac{n}{2} + d)$. The investigation of the Riemann-Hilbert problem for the sNLS equation yielding this estimate will be published elsewhere.

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